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An algebraic construction of generalized coherent states for shape-invariant potentials

A N F Aleixo[1] **and A B Balantekin**[2]

[1] Instituto de Física, Universidade Federal do Rio de Janeiro, RJ, Brazil
[2] Department of Physics, University of Wisconsin, Madison, WI 53706, USA

E-mail: armando@if.ufrj.br and baha@nucth.physics.wisc.edu

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Abstract

Generalized coherent states for shape-invariant potentials are constructed using an algebraic approach based on supersymmetric quantum mechanics. We show that this generalized formalism is able to (a) supply the essential requirements necessary to establish a connection between classical and quantum formulations of a given system (continuity of labelling, resolution of unity, temporal stability and action identity), (b) reproduce results already known for shape-invariant systems, such as harmonic oscillator, double anharmonic, Pöschl–Teller and self-similar potentials, and (c) point to a formalism that provides a unified description of the different kind of coherent states for quantum systems.

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1. Introduction

Coherent states were first introduced by Schrödinger [1], who was interested in finding quantum-mechanical states which provide a close connection between quantum and classical formulations of a given physical system. Based on the Heisenberg–Weyl group and applied specifically to the harmonic oscillator system, the original coherent state introduced by Schrödinger has been extended to a large number of Lie groups with square integrable representations [2, 3]. Today these extensions represent many applications in a number of fields of quantum theory, and especially in quantum optics and radiophysics. In particular they are used as bases of coherent state path integrals [4] or dynamical wavepackets for describing the quantum systems in semi-classical approximations [5]. There are different definitions of coherent states. The first one, often called Barut–Girardello coherent states [6], assumes that the coherent states are eigenstates with complex eigenvalues of an annihilation group operator. The second definition, often called Perelomov coherent states [7], assumes the existence of a unitary z-displacement operator whose action on the ground state of the system gives the

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coherent state parametrized by z, with $z \in \mathbb{C}$. The last definition, based on the Heisenberg uncertainty relation, often called *intelligent* coherent states [8], assumes that the coherent state gives the minimum-uncertainty value $\Delta x \Delta p = \frac{\hbar}{2}$, and maintains this relation in time because of its temporal stability. These three different definitions are equivalent only in the special case of the Heisenberg–Weyl group, the dynamical symmetry group of the harmonic oscillator.

The extension of coherent states for systems other than the harmonic oscillator has attracted much attention for the past several years [9–15]. There are a large number of different approaches to this problem and the one to be presented here is based on the supersymmetric quantum mechanics. Supersymmetric quantum mechanics [16] deals with pairs of Hamiltonians which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [17]. Although not all exactly-solvable problems are shape-invariant [18], shape invariance, especially in its algebraic formulation [19, 20], is a powerful technique to study exactly-solvable systems.

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x) = \hbar\Omega\hat{A}^\dagger\hat{A} \qquad \text{and} \qquad \hat{H}_2 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x) = \hbar\Omega\hat{A}\hat{A}^\dagger \tag{1.1}$$

are most readily written in terms of one-dimensional operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p}\right) \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p}\right) \tag{1.2}$$

where $\hbar\Omega$ is a constant energy scale factor, introduced to permit working with dimensionless quantities, and $W(x)$ is the superpotential which is related to the potentials $V_\pm(x)$ via

$$V_\pm(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{1.3}$$

In earlier papers, by using an algebraic approach, we introduced coherent states for self-similar potentials [13] and a class of shape-invariant systems and presented a possible generalization of these coherent states and its relation with Ramanujan's integrals [14]. In the present paper we extend this generalized formalism to all shape-invariant systems and show that the generalized coherent states then obtained satisfy the essential requirements necessary to provide the basic principles [21] embodied in Schrödinger's original idea. This paper is organized as follows. In section 2 we present the algebraic formulation of shape invariance and introduce the fundamental principles of our generalized coherent states and their basic properties; in section 3 we apply our general formalism to shape-invariant systems classified using the factorization method introduced by Infeld and Hull [22] and work out some possible examples of coherent states for these systems. Finally, brief remarks close the paper in section 4.

2. Generalized coherent states for shape-invariant systems

The Hamiltonian $\hat{H}_1$ of equation (1.1) is called shape invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1), \tag{2.1}$$

is satisfied [17]. In this equation a_1 and a_2 represent the parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As it is written the condition of equation (2.1) does not require the Hamiltonian to be one dimensional, and one does not need to choose the ansatz of equation (1.2). In the cases studied so far the parameters a_1 and a_2 are either related by a translation [19, 23] or a scaling [13, 14, 24]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator $\hat{T}(a_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2)$ and the operators $\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1)$ and $\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1)$, the Hamiltonians of equation (1.1) take the forms $\hat{H} \equiv \hat{H}_1 = \hbar\Omega\hat{B}_+\hat{B}_-$ and $\hat{H}_2 = \hbar\Omega\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger$. As shown in [19], with equation (2.1) one can also easily prove the commutation relation $[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0)$, where we used the identity $R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1)$, valid for any $n \in \mathbb{Z}$. This commutation relation suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with $R(a_1)$ is taken into account. The additional relations

$$R(a_n)\hat{B}_+ = \hat{B}_+R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_-R(a_{n+1}), \tag{2.2}$$

readily follow from these results. Considering that the ground state of the Hamiltonian $\hat{H}$ satisfies the condition

$$\hat{A}|\Psi_0\rangle = 0 = \hat{B}_-|\Psi_0\rangle, \tag{2.3}$$

using the relations above it is possible to find the nth excited state of $\hat{H}$

$$\hat{H}|\Psi_n\rangle \equiv \hbar\Omega(\hat{B}_+\hat{B}_-)|\Psi_n\rangle = \hbar\Omega e_n|\Psi_n\rangle \qquad \text{and} \qquad \hat{B}_-\hat{B}_+|\Psi_n\rangle = \{e_n + R(a_0)\}|\Psi_n\rangle. \tag{2.4}$$

where these eigenstates can be written in a normalized form as

$$|\Psi_n\rangle = \frac{1}{\sqrt{R(a_1) + R(a_2) + \cdots + R(a_n)}}\hat{B}_+ \cdots \frac{1}{\sqrt{R(a_1) + R(a_2)}}\hat{B}_+\frac{1}{\sqrt{R(a_1)}}\hat{B}_+|\Psi_0\rangle \tag{2.5}$$

with the eigenvalues $E_n = \hbar\Omega e_n$, being

$$e_n = \sum_{k=1}^{n} R(a_k). \tag{2.6}$$

As mentioned in the introduction, a possible way to define a coherent state is to find a quantum state annihilated by the lowering operator. Annihilation-operator coherent states for shape-invariant potentials were introduced in [10, 13]. Here we follow the notation of [13]. Our first step is to introduce the necessary tools to be used in this construction. After we obtain the coherent state we must verify if this state satisfies the set of four essential requirements, introduced and discussed in [21], necessary for a close connection between classical and quantum formulations of a given system: (a) label continuity, (b) overcompleteness or resolution of unity, (c) temporal stability and (d) action identity. Indeed the first two requirements are standard and rely on the algebraic structure behind the system in question, while the last two are more general and relate to the classical-quantum connection question.

2.1. Construction

To remove the energy scale we rewrite the shape-invariant Hamiltonian as

$$\hat{H} = \hbar\Omega\hat{\mathcal{H}}, \qquad \text{with} \quad \hat{\mathcal{H}} = \hat{B}_+\hat{B}_-. \tag{2.7}$$

The operator $\hat{B}_-$ does not have a left inverse in the Hilbert space of the eigenstates of the Hamiltonian $\hat{H}$. However, a right inverse for $\hat{B}_-$ ($\hat{B}_-\hat{B}_-^{-1} = \hat{1}$) can be defined. Similarly the inverse of $\hat{\mathcal{H}}$ does not exist, but $\hat{\mathcal{H}}^{-1}\hat{B}_+ = \hat{B}_-^{-1}$ does. Therefore, if we define the

Hermitian conjugate operators $\hat{Q} = \hat{B}_-\hat{\mathcal{H}}^{-1/2}$ and $\hat{Q}^\dagger = \hat{\mathcal{H}}^{-1/2}\hat{B}_+$, we can easily show that $\hat{B}_-^{-1} = \hat{\mathcal{H}}^{-1/2}\hat{Q}^\dagger$ and the normalized form of the nth excited state of $\hat{H}$, given by (2.5), can be rewritten as $|\Psi_n\rangle = (\hat{Q}^\dagger)^n|\Psi_0\rangle$. Then, taking into account equations (2.4) and (2.6) and these last two relations, we can prove that

$$\hat{B}_-^{-n}|\Psi_0\rangle = C_n|\Psi_n\rangle, \qquad \text{where}$$
$$C_n = \left\{\prod_{k=0}^{n-1}(e_n - e_k)\right\}^{-1/2} = \left\{\prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right]\right\}^{-1/2}, \tag{2.8}$$

since $e_0 = 0$. After these preliminary considerations we are ready to define our generalized expression for the coherent state of shape-invariant systems as

$$|z; a_j\rangle = \sum_{n=0}^{\infty}\{z\mathcal{Z}_j\hat{B}_-^{-1}\}^n|\Psi_0\rangle, \qquad z, \mathcal{Z}_j \in \mathbb{C}, \tag{2.9}$$

where we used the shorthand notation $\mathcal{Z}_j \equiv \mathcal{Z}(a_j) \equiv \mathcal{Z}(a_1, a_2, a_3, \ldots)$ for an arbitrary functional of the potential parameters, introduced to establish a more general approach. As one will see in the applications below, for harmonic oscillator system, the presence of the functional $\mathcal{Z}_j$ introduces only a constant scale factor in the complex expansion variable z that can be absorbed by a redefinition of this constant and, thus, we get back to the standard results for this system. Formally the definition (2.9) can be expressed as

$$|z; a_j\rangle = \left[\frac{1}{1 - z\mathcal{Z}_j\hat{B}_-^{-1}}\right]|\Psi_0\rangle. \tag{2.10}$$

Using relation (2.2) we can prove that this coherent state is an eigenstate of the operator $\hat{B}_-$ since

$$\hat{B}_-|z; a_j\rangle = z\mathcal{Z}_{j-1}|z; a_j\rangle. \tag{2.11}$$

This state also satisfies the additional condition

$$\{\hat{B}_- - z\mathcal{Z}_{j-1}\}\frac{\partial}{\partial z}|z; a_j\rangle = \mathcal{Z}_{j-1}|z; a_j\rangle, \tag{2.12}$$

where $\mathcal{Z}_{j-1} = \hat{T}^\dagger(a_1)\mathcal{Z}_j\hat{T}(a_1)$. An important observation is that the coherent state definition (2.9) satisfies the continuity of labelling requirements since the transformation of the variables $(z, a_j) \to (z', a_{j'})$ leads to the transformation of the states $|z; a_j\rangle \to |z'; a_{j'}\rangle$. This is the first standard property required for coherent states. The other three we take up in the next subsections. Together with the resolution of unity, the continuity of labelling represents the minimal condition to be satisfied for a set of coherent states to be represented by a Lie algebraic group.

2.2. Normalization

At this stage we can use the action of the $\hat{B}_-^{-1}$ operator on the Hilbert space of the eigenstates $\{|\Psi_n\rangle, n = 0, 1, 2, \ldots\}$ and (2.2) to get the generalized Glauber's form [25] of the coherent state $|z; a_j\rangle$ based in its expansion in the eigenstates of the Hamiltonian $\hat{H}$:

$$|z; a_r\rangle = \mathcal{N}(|z|^2; a_r)\sum_{n=0}^{\infty}\frac{z^n}{h_n(a_r)}|\Psi_n\rangle, \tag{2.13}$$

where we used the shorthand notation $(a_r) \equiv [R(a_1), R(a_2), \ldots, R(a_n); a_j, a_{j+1}, \ldots, a_{j+n-1}]$ for the expansion coefficients, which are given by $h_0(a_r) = 1$ and

$$h_n(a_r) = \sqrt{\prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right]} \Bigg/ \prod_{k=0}^{n-1} \mathcal{Z}_{j+k}, \qquad \text{for} \quad n \geqslant 1 \tag{2.14}$$

with $\mathcal{Z}_{j+k} = \{\hat{T}(a_1)\}^k \mathcal{Z}_j \{\hat{T}^\dagger(a_1)\}^k$, as well as for the real normalization factor

$$\mathcal{N}(x; a_r) = 1 \Bigg/ \sqrt{\sum_{n=0}^{\infty} \frac{x^n}{|h_n(a_r)|^2}}. \tag{2.15}$$

At this point we observe that the transformation properties between the potential parameters a_n, imposed by shape invariance, constrain the freedom to define $\mathcal{Z}_j$. Besides that, when we consider relation (2.14), this potential parameter dependence in $\mathcal{Z}_j$ shows strong influence in the final expression of the expansion coefficient $h_n(a_r)$. Another thing to observe about $\mathcal{Z}_j$ is its importance in the determination of the radius of convergence in the series defining $\mathcal{N}(|z|^2; a_r)$ since this radius is given by $\mathcal{R} = \limsup_{n\to+\infty} \sqrt[n]{|h_n(a_r)|^2}$.

It should be noted that this normalized coherent state has a $\hat{B}_-$ operator eigenvalue different from the unnormalized one since the potential parameters in the normalization factor are changed by the action of that operator. Indeed we can prove that in this case, equation (2.11) must assume the form

$$\hat{B}_-|z; a_j\rangle = z\mathcal{Z}_{j-1}\left[\frac{\mathcal{N}(a_{r-1}; |z|^2)}{\mathcal{N}(a_r; |z|^2)}\right]|z; a_j\rangle, \tag{2.16}$$

where $\mathcal{N}(a_{r-1}; |z|^2) = \hat{T}^\dagger(a_1)\mathcal{N}(a_r; |z|^2)\hat{T}(a_1)$. Although they are normalized, the coherent states $|z; a_r\rangle$ are not orthogonal to each other since

$$\langle z'; a_r|z; a_r\rangle = \frac{\mathcal{N}(a_r; |z'|^2)\mathcal{N}(a_r; |z|^2)}{\mathcal{N}^2(a_r; zz'^*)}. \tag{2.17}$$

So we conclude that they form an over-complete linearly dependent set.

2.3. Overcompleteness

Now we can investigate the overcompleteness or resolution of unity property of the generalized coherent states introduced by equation (2.9). To this end we assume the existence of a positive-definite weight function $w(|z|^2; a_r)$ so that an integral over the complex plane exists and gives the result

$$\int_{\mathbb{C}} \mathrm{d}^2 z |z; a_r\rangle\langle z; a_r| w(|z|^2; a_r) = \hat{1}_{\mathcal{H}}, \tag{2.18}$$

where $\hat{1}_{\mathcal{H}}$ is the identity operator in the Hilbert space of the $\hat{H}$-eigenstates. Inserting equation (2.13) into equation (2.18) the resolution of unity can be expressed by

$$\int_{\mathbb{C}} \mathrm{d}^2 z\, \mathcal{N}^2(|z|^2; a_r) \sum_{m,n=0}^{\infty} \frac{z^{*m} z^n}{h_m^*(a_r) h_n(a_r)} |\Psi_n\rangle\langle\Psi_m| w(|z|^2; a_r) = \hat{1}_{\mathcal{H}}. \tag{2.19}$$

At this point we can use the orthonormality of the eigenstates $|\Psi_n\rangle$ to show that the diagonal matrix elements of equation (2.19) can be written as

$$\int_{\mathbb{C}} \mathrm{d}^2 z\, \mathcal{N}^2(|z|^2; a_r)(z^* z)^n w(|z|^2; a_r) = |h_n(a_r)|^2. \tag{2.20}$$

Therefore, assuming the polar coordinate representation $z \equiv r\,\mathrm{e}^{\mathrm{i}\phi}$ of complex numbers we must have $\mathrm{d}^2 z = r\,\mathrm{d}r\,\mathrm{d}\phi$ and using the result

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\,\mathrm{e}^{\mathrm{i}(n-m)\phi} = \delta_{n,m}, \tag{2.21}$$

we conclude that to get a resolution of unity we must require

$$\int_0^\infty \mathrm{d}\rho\,\rho^n \mathcal{W}(\rho; a_r) = |h_n(a_r)|^2, \qquad \text{where} \qquad \mathcal{W}(\rho; a_r) = \pi\mathcal{N}^2(\rho; a_r) w(\rho; a_r), \tag{2.22}$$

and ρ stands for r^2. In other words, equation (2.22) provides the set of moments $\{\rho_n\}$ of the distribution function $\mathcal{W}(\rho; a_r)$, since we assume all moments exist and have finite values. Therefore, as pointed out in [26], the problem of finding a suitable measure $w(\rho; a_r)$ reduces to a moment distribution problem. After this point there are several possible ways to get the measure $w(\rho; a_r)$. We can choose a possible form of $w(\rho; a_r)$ by using the result of a known integral. Another possibility is to use a transformation procedure, like Mellin [27] or Fourier, to determine the form of the measure $w(\rho; a_r)$. For example, in the Fourier transformation case we can multiply equation (2.22) by the sum factor $\sum_{n=0}^{\infty}(\mathrm{i}\xi)^n/n!$ and use the series expansion of the exponential function to obtain

$$\int_0^\infty \mathrm{d}\rho\,\mathcal{W}(\rho; a_r)\,\mathrm{e}^{\mathrm{i}\rho\xi} = \Phi(\xi; a_r) = \sum_{n=0}^{\infty} |h_n(a_r)|^2 (\mathrm{i}\xi)^n/n!. \tag{2.23}$$

Thus, taking the inverse Fourier transformation of equation (2.23) we can show that

$$\mathcal{W}(\rho; a_r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi\;\Phi(\xi; a_r)\,\mathrm{e}^{-\mathrm{i}\rho\xi}. \tag{2.24}$$

In the applications of the next section we will use several different procedures to get the resolution of unity. To conclude this part, we note that explicit computation of the weight function $w(|z|^2; a_r)$ requires the knowledge of the spectrum of the quantum mechanical system under consideration and the form of the functional $\mathcal{Z}_j$.

2.4. Temporal stability

Let us now investigate the dynamical evolution of the generalized coherent state (2.13). To do that we must remember that the time evolution of this generalized coherent state can be obtained by

$$|z; a_r\rangle_t = \hat{U}(t, 0)|z; a_r\rangle_o \tag{2.25}$$

where the time evolution operator fulfils the differential equation

$$\mathrm{i}\hbar\frac{\partial \hat{U}(t, 0)}{\partial t} = \hat{H}\hat{U}(t, 0), \tag{2.26}$$

with the initial condition $\hat{U}(0, 0) = \hat{\mathbb{1}}_{\mathcal{H}}$. Thus,

$$|z; a_r\rangle_t = \exp(-\mathrm{i}\hat{H}t/\hbar)|z; a_r\rangle_o. \tag{2.27}$$

At this point if we consider expansion (2.13), the results of equations (2.4) and the commutation between any function of the potential parameters a and the Hamiltonian $\hat{H}$ in equation (2.27) we obtain

$$|z; a_r\rangle_t = \mathcal{N}(|z|^2; a_r)\sum_{n=0}^{\infty}\frac{z^n}{h_n(a_r)}\,\mathrm{e}^{-\mathrm{i}\Omega e_n t}|\Psi_n\rangle. \tag{2.28}$$

To establish the temporal stability of this coherent state we utilize the freedom in the choice of the functional $\mathcal{Z}(a_j)$ to redefine it as $\bar{\mathcal{Z}}(a_j) = \mathcal{Z}(a_j)\,\mathrm{e}^{-\mathrm{i}\alpha R(a_1)}$ where α is a real constant. This redefinition implies $\bar{h}_n(a_r) = h_n(a_r)\,\mathrm{e}^{\mathrm{i}\alpha e_n}$, where e_n is given by (2.6) and $h_n(a_r)$ still given by equation (2.14). Therefore we can write the coherent state $|z; a_r\rangle$ as

$$|z; a_r\rangle \Longrightarrow |z, \alpha; a_r\rangle = \mathcal{N}(|z|^2; a_r) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)}\,\mathrm{e}^{-\mathrm{i}\alpha e_n} |\Psi_n\rangle, \tag{2.29}$$

and its time-evolved form as

$$|z, \alpha; a_r\rangle_t = \mathcal{N}(|z|^2; a_r) \sum_{n=0}^{\infty} \frac{z^n}{h_n(a_r)}\,\mathrm{e}^{-\mathrm{i}(\alpha+\Omega t)e_n} |\Psi_n\rangle \equiv |z, \alpha + \Omega t; a_r\rangle, \tag{2.30}$$

illustrating the fact that the time evolution of any such generalized coherent state remains within the family of generalized coherent states. In other words, the generalized coherent states $|z, \alpha; a_r\rangle$ show temporal stability under $\hat{H}$. To conclude this part, note that the polar coordinates representation of the redefined complex functional $\bar{\mathcal{Z}}(a_j)$ implies that in the coherent state time evolution its real modulus remains constant while its complex phase increases linearly. These properties are similar to the classical behaviour of canonical action-angle variables.

2.5. Action identity

The last property to be satisfied for the coherent state $|z; a_r\rangle$ is the action identity. To verify this identity we take the conjugate of equation (2.11) and use the definition of the operator $\hat{B}_+$ to get

$$\langle z; a_r|\hat{B}_+ = \langle z; a_r| z^* \mathcal{Z}_{j-1}^*. \tag{2.31}$$

Now with this result, equation (2.11) and the expression of the Hamiltonian $\hat{H}$ we can calculate the expectation value

$$\langle \hat{H}\rangle = \frac{\langle z; a_r|\hat{H}|z; a_r\rangle}{\langle z; a_r|z; a_r\rangle} = \hbar\Omega \frac{\langle z; a_r|\hat{B}_+\hat{B}_-|z; a_r\rangle}{\langle z; a_r|z; a_r\rangle} = \hbar\Omega |z\mathcal{Z}_{j-1}|^2. \tag{2.32}$$

Using this result we can define a canonical action variable $J = \hbar\beta_j^*\beta_j$, with $\beta_j = z\mathcal{Z}_{j-1}$, such as $\langle\hat{H}\rangle = \nu J$, so that $\dot{\nu} = \partial\langle\hat{H}\rangle/\partial J = \Omega \Longrightarrow \nu = \Omega t + \alpha$, as required for a couple of canonical conjugate action-angle variables. Note that the normalized form (2.13) of the coherent state $|z; a_r\rangle$ requires the definition $\beta_j = z\mathcal{Z}_{j-1}\mathcal{N}(|z|^2; a_{r-1})/\mathcal{N}(|z|^2; a_r)$.

With these properties we showed that the generalized coherent state $|z; a_r\rangle$ satisfies the set of basic requirements we enumerated.

3. Some examples of generalized coherent state systems

Using the definition presented in the previous section we now illustrate the concept of generalized coherent states for shape-invariant systems using some known shape-invariant potential systems. As in [10], for these applications we follow the classification based on the factorization method introduced by Infeld and Hull [22] in which six possibles types of shape-invariant systems are grouped when their potential parameters are related by a translation. We also study the case of the self-similar potential system as an example of shape-invariant potential with potential parameters related by scaling.

3.1. Types (C) and (D) shape-invariant systems

We begin with these systems because they are the simplest cases among the shape-invariant potential systems. The partner potentials $V_\pm(x)$ for these systems are obtained with the superpotentials

$$W_C(x, a_1) = \sqrt{\hbar\Omega}\left(\frac{a_1+\delta}{x} + \frac{\beta}{2}x\right), \qquad \text{and} \qquad W_D(x, a_1) = \sqrt{\hbar\Omega}(\beta x + \delta), \tag{3.1}$$

where β and δ are real constants, while the remainders in the shape-invariant condition (2.1) are given by [16]

$$R_C(a_n) = \beta\left(a_n - a_{n+1} + \sqrt{\frac{\hbar}{2m\Omega}}\right), \qquad \text{and} \qquad R_D(a_n) = \sqrt{\frac{\hbar}{2m\Omega}}(a_n + a_{n+1}). \tag{3.2}$$

Taking into account that the parameters for these potentials are related by

$$\begin{cases} a_{n+1} = a_n - \sqrt{\hbar/(2m\Omega)}, & \text{for (C)}, \\ a_1 = a_2 = \cdots = a_n = \beta, & \text{for (D)}, \end{cases} \qquad (\forall n \in \mathbb{Z}), \tag{3.3}$$

we conclude that for both shape-invariant systems the remainders (3.2) can be written as $R(a_n) = \gamma$, with $\gamma = \sqrt{2\hbar/(m\Omega)}\beta$, and thus

$$\prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right] = \prod_{k=1}^{n}[\gamma(n-k+1)] = \gamma^n n!. \tag{3.4}$$

On the other hand, the constant values of the potential parameters for (D) shape-invariant potential imply that for these systems we must have $\mathcal{Z}_j = c$, a constant. Using this and equation (2.14) we obtain

$$\prod_{k=0}^{n-1}\mathcal{Z}_{j+k} = c^n \quad \Longrightarrow \quad h_n(a_r) = \frac{\sqrt{\gamma^n n!}}{c^n}. \tag{3.5}$$

Taking this into account in equations (2.15) and (2.13) we find

$$\mathcal{N}(|z|^2; a_r) = \exp\left(-\frac{c^2|z|^2}{2\gamma}\right), \qquad \text{and} \qquad |z; a_r\rangle = \mathrm{e}^{-(c|z|/\sqrt{2\gamma})^2}\sum_{n=0}^{\infty}\frac{(cz/\sqrt{\gamma})^n}{\sqrt{n!}}|\Psi_n\rangle. \tag{3.6}$$

With these results we can show that the inner product (2.17) of two coherent states can be readily found as

$$\langle z'; a_r|z; a_r\rangle = \exp\left[-\frac{c^2}{2\gamma}(|z'|^2 + |z|^2 - 2zz'^*)\right]. \tag{3.7}$$

The overcompleteness property can be verified by using equation (2.23). The function

$$\Phi(\xi; a_r) = \sum_{n=0}^{\infty}\left(\frac{\mathrm{i}\gamma\xi}{c^2}\right)^n = \frac{1}{1 - \mathrm{i}\gamma\xi/c^2}, \tag{3.8}$$

which has a pole at $\xi = -\mathrm{i}c^2/\gamma$ and its integration in equation (2.24) by using the lower-half complex plane enclosing this pole yields

$$\mathcal{W}(\rho; a_r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{\mathrm{e}^{-\mathrm{i}\rho\xi}}{1 - \mathrm{i}\gamma\xi/c^2} = \mathrm{e}^{-c^2\rho/\gamma}. \tag{3.9}$$

Now, taking into account the result for $\mathcal{N}(\rho; a_r)$ and the relation between the function $\mathcal{W}(\rho; a_r)$ and the weight function, it is possible to show that $w(\rho; a_r) = 1/\pi$. The example of a (D) shape-invariant system is the harmonic oscillator potential $V_-(x, a_1)$, obtained with equation (1.3) and $W_D(x, a_1)$. In this case it should be noted that if we redefine the complex constant by $z \longrightarrow cz/\sqrt{\gamma}$, and take into account that $|\Psi_n\rangle \rightarrow |n\rangle$ is an element of the Fock space $\{|n\rangle, n \geqslant 0\}$, we obtain for the coherent state and its inner product

$$|z; a_r\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \qquad \langle z'; a_r | z; a_r\rangle = \exp\left[-\frac{1}{2}(|z'|^2 + |z|^2 - 2zz'^*)\right], \tag{3.10}$$

which are the usual expressions for bosonic coherent states [3]. In this case we observe that the simplicity of the harmonic oscillator system does not permit any special modification in the standard result with the definition of the generalized coherent state (2.13).

For (C) shape-invariant systems, if we make the choice $\mathcal{Z}_j = c$ (a constant) and following the steps above it is possible to obtain identical results as (D) shape-invariant systems for the coherent state. However, any other choice would imply different results. Just as an example, let us define the following auxiliary function,

$$g(a_j; c, d) = ca_j + d, \tag{3.11}$$

where c and d are constants. With the help of equation (3.3) we can show that

$$\prod_{k=0}^{n-1} g(a_{j+k}; c, d) = \frac{(-c\eta)^n \Gamma[n + j - \rho - \mathrm{d}/(c\eta) - 1]}{\Gamma[j - \rho - \mathrm{d}/(c\eta) - 1]} = \frac{(c\eta)^n \Gamma[\rho + \mathrm{d}/(c\eta) - j + 2]}{\Gamma[\rho + \mathrm{d}/(c\eta) - n - j + 2]}, \tag{3.12}$$

where $\eta = \sqrt{\hbar/(2m\Omega)}$ and $\rho = a_1/\eta$. Taking into account this result and defining the functional $\mathcal{Z}_j$ as

$$\mathcal{Z}_j = \sqrt{g(a_1; -\gamma/\eta, 1)}\, \mathrm{e}^{-\mathrm{i}\alpha R(a_1)} \tag{3.13}$$

we get

$$\prod_{k=0}^{n-1} \mathcal{Z}_{j+k} = \sqrt{\frac{\gamma^n \Gamma(n - \rho)}{\Gamma(-\rho)}}\, \mathrm{e}^{-\mathrm{i}\alpha\gamma n}, \tag{3.14}$$

where we used that $e_n = n\gamma$. Substituting equations (3.4) and (3.14) in (2.14) we obtain

$$h_n(a_r) = \sqrt{\frac{\Gamma(-\rho)\Gamma(n + 1)}{\Gamma(n - \rho)}}\, \mathrm{e}^{\mathrm{i}\alpha\gamma n}, \tag{3.15}$$

and we can show that the normalization factor (2.15) in this case is given by

$$\mathcal{N}(|z|^2; a_r) = \left[\frac{1}{\Gamma(-\rho)} \sum_{n=0}^{\infty} \frac{\Gamma(n - \rho)}{\Gamma(n + 1)} |z|^{2n}\right]^{-1/2} = (1 - |z|^2)^{-\rho/2}, \tag{3.16}$$

with the restriction $|z| < 1$. Thus, the coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = (1 - |z|^2)^{-\rho} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n - \rho)}{\Gamma(-\rho)\Gamma(n + 1)}}\, \mathrm{e}^{-\mathrm{i}\alpha\gamma n} z^n |\Psi_n\rangle, \tag{3.17}$$

where we take $a_1 < 0$ implying $\rho < 0$. As it is always possible to get $a_1 + \delta > 0$ with an adequate choice of δ, there are no problems with this assumption. In this case the inner product (2.17) of two coherent states will be

$$\langle z'; a_r | z; a_r\rangle = \left[\frac{\sqrt{(1 - |z|^2)(1 - |z'|^2)}}{(1 - z'^* z)}\right]^{-\rho}. \tag{3.18}$$

The completeness can be obtained by using the measure $w(|z|^2; a_r) = -(\rho+1)(1-|z|^2)^{-2}/\pi$, that is invariant on the disc $|z| < 1$. Example of a shape-invariant type (C) system [28] is the double anharmonic potential $V_-(x, a_1)$, obtained with equation (1.3) and using the superpotential $W_C(x, a_1)$.

The coherent state we obtained for the (C)-type shape-invariant system, equation (3.17), is the Perelomov coherent state [7] for the group $SU(1, 1)$. This is not surprising since the $SU(1, 1)$ algebra is both the shape-invariance and spectrum-generating algebra of this shape-invariant system. The appropriate realization of this algebra is

$$\hat{K}_0 = \frac{1}{4}\left(\hat{p}^2 + x^2 + \frac{\alpha}{x^2}\right) \qquad \text{and} \qquad \hat{K}_\pm = \frac{1}{4}\left(\hat{p}^2 - x^2 + \frac{\alpha}{x^2}\right) \pm \frac{\mathrm{i}}{4}\left(\hat{p}x + x\hat{p}\right). \tag{3.19}$$

For the (C)-type shape-invariant systems the shape invariance [19] connects eigenstates of the same system.

3.2. Types (A) and (B) shape-invariant systems

The partner potentials $V_\pm(x)$ for these systems are obtained with the superpotentials

$$W_A(x, a_1) = \sqrt{\hbar\Omega}\{\beta(a_1 + \gamma)\cot[\beta(x + \lambda)] + \delta\csc[\beta(x + \lambda)]\} \tag{3.20}$$

$$W_B(x, a_1) = \sqrt{\hbar\Omega}[\beta(a_1 + \gamma) + \delta\exp(-\beta x)], \tag{3.21}$$

β, γ, δ and λ being real constants. For these systems the remainders in the shape-invariant condition (2.1) are given by $R(a_1) = \pm\beta^2\eta[2(a_1 + \gamma) \pm \eta]$, with the potential parameters related by $a_{n+1} = a_n \pm \eta$, where $\eta = \sqrt{\hbar/(2m\Omega)}$ and the signs (+) and (−) stand for (A) and (B) types, respectively. Using these results we can prove that for (A)-type systems one has

$$\prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right] = \frac{\kappa^{2n}\Gamma(n + 1)\Gamma(2\rho + 2n)}{\Gamma(2\rho + n)}, \tag{3.22}$$

with $\kappa = \eta\beta$ and $\rho = (a_1 + \gamma)/\eta$. To investigate the consequences of our general approach for this type of system let us consider some possibilities. First, if we make the choice $\mathcal{Z}_j = c$ (a constant) and use the result of equations (3.5) and (3.22) we find

$$h_n(a_r) = \sqrt{\frac{\Gamma(n + 1)\Gamma(2\rho + 2n)}{\Gamma(2\rho + n)}} \tag{3.23}$$

and

$$\mathcal{N}(|z|^2; a_r) = \left[\sum_{n=0}^{\infty} \frac{\Gamma(2\rho + n)}{\Gamma(n + 1)\Gamma(2\rho + 2n)}|z|^{2n}\right]^{-1/2} \tag{3.24}$$

for the expansion coefficient and the normalization factor, respectively, after assuming $c = \kappa$. At this point, if we choose $\rho = 1/2$ we get the simple expression found in [10] for the coherent state (2.13) because in this case $h_n(a_r) = \sqrt{(2n)!}$, and since by equation (2.15)

$$\mathcal{N}(|z|^2; a_r)^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!}} = \frac{1}{\sqrt{\mathrm{sech}(|z|)}}, \qquad \text{and}$$

$$|z; a_r\rangle = \sqrt{\mathrm{sech}(|z|)}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2n)!}}|\Psi_n\rangle. \tag{3.25}$$

As shown in [10], in this case the identity resolution is obtained with the measure $w(|z|^2; a_r) = \mathrm{e}^{-|z|}/(2|z|)$.

Another interesting possibility is to use the auxiliary function (3.11). In this case, because of the translation relation between the a_n potential parameters, we can prove that for type (A) systems one gets

$$\prod_{k=0}^{n-1} g(a_{j+k}; c, d) = \frac{(c\eta)^n \Gamma\left[\frac{\nu}{2} + j + n + \mathrm{d}\big/(c\eta) - 1\right]}{\Gamma\left[\frac{\nu}{2} + j + \mathrm{d}\big/(c\eta) - 1\right]}, \tag{3.26}$$

where $\nu = 2a_1/\eta$. If we define the functional $\mathcal{Z}_j = \sqrt{g(a_1; 2\kappa/\eta, \kappa) g(a_1; 2\kappa/\eta, 2\kappa)}\, \mathrm{e}^{-\mathrm{i}\alpha R(a_1)}$, we obtain

$$\prod_{k=0}^{n-1} \mathcal{Z}_{j+k} = \sqrt{\frac{(2\kappa)^{2n}\Gamma\left(\frac{\nu}{2}+n+1\right)\Gamma\left(\frac{\nu}{2}+n+\frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}}\, \mathrm{e}^{-\mathrm{i}\alpha e_n} = \sqrt{\frac{\kappa^{2n}\Gamma(\nu+2n+1)}{\Gamma(\nu+1)}}\, \mathrm{e}^{-\mathrm{i}\alpha e_n}, \tag{3.27}$$

where $e_n = \kappa^2 n(n+2\rho)$. Assuming $\gamma = \eta/2$ and using equations (3.22) and (3.27) in (2.14) we obtain

$$h_n(a_r) = \sqrt{\frac{\Gamma(\nu+1)\Gamma(n+1)}{\Gamma(\nu+n+1)}}\, \mathrm{e}^{\mathrm{i}\alpha e_n}, \tag{3.28}$$

and we can show that the normalization factor (2.15) in this case is given by

$$\mathcal{N}(|z|^2; a_r) = \left[\frac{1}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n+1)}{\Gamma(n+1)} |z|^{2n}\right]^{-1/2} = (1-|z|^2)^{(\nu+1)/2}, \tag{3.29}$$

with the restriction $|z| < 1$. The coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = (1-|z|^2)^{(\nu+1)/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)\Gamma(n+1)}}\, \mathrm{e}^{-\mathrm{i}\alpha e_n} z^n |\Psi_n\rangle. \tag{3.30}$$

Comparing with [29] one notes that equation (3.30) is a form for the coherent state of the Pöschl–Teller potential of first type [30]. This potential $V_-(x, a_1)$, obtained with equation (1.3) and using the superpotential $W_{\mathrm{A}}(x, a_1)$, is the example of a shape-invariant system type (A). As shown in [29], in this case, the resolution of unity is obtained with the measure $w(|z|^2; a_r) = \nu(1-|z|^2)^{-2}/\pi$.

Finally, note that if one takes

$$\mathcal{Z}_j = \frac{\sqrt{g(a_1; 2/\eta, 1) g(a_1; 2/\eta, 2)}}{g[a_1; 1/(\kappa\eta), (1+\nu/2)/\kappa]}\, \mathrm{e}^{-\mathrm{i}\alpha R(a_1)}, \tag{3.31}$$

and follows the same way used before one gets a second possible form for the coherent state of the Pöschl–Teller potential [29, 31]

$$|z; a_r\rangle = \frac{|z|^{\nu/2}}{\sqrt{I_\nu(2|z|)}} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\alpha e_n} z^n}{\sqrt{\Gamma(n+1)\Gamma(\nu+n+1)}} |\Psi_n\rangle, \tag{3.32}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. As shown in [29, 31], in this case the resolution of unity is given by the measure

$$w(|z|^2; a_r) = \frac{2}{\pi} K_\nu(2|z|^2), \quad \text{with} \quad K_\nu(x) = \frac{\pi[I_{-\nu}(x) - I_\nu(x)]}{2\sin(\pi\nu)}, \qquad \nu \notin \mathbb{Z}. \tag{3.33}$$

Note that the coherent state in equation (3.32) is the Barut–Girardello coherent state for the $SU(1, 1)$ algebra [6]. This is not surprising since $SU(1, 1)$ is the shape-invariance algebra for the Pöschl–Teller potential as shown in [19]. Note that in this case the shape-invariant potential relates a series of potentials with different depths, not the quantum states of the given potential, i.e. the shape-invariance algebra is not the spectrum-generating algebra in contrast to the (C)- and (D)-type shape-invariant systems. Hence the coherent state corresponds to a non-compact algebra with infinite number of states representing all possible potentials with different depths.

As a last example we obtain a new coherent state for this kind of system with the introduction of the functional

$$\mathcal{Z}_j = \sqrt{\frac{g(a_1;\beta,\beta\gamma)g(a_1;\beta,\beta\gamma+\kappa/2)g(a_3;2/\eta,-\nu-2\sigma)}{g(a_1;1/\eta,\rho+\gamma/\eta)g(a_3;1/\eta,-\nu/2)}}\,\mathrm{e}^{-\mathrm{i}\alpha R(a_1)}, \tag{3.34}$$

which leads to

$$\prod_{k=0}^{n-1}\mathcal{Z}_{j+k} = \sqrt{\frac{\kappa^{2n}\Gamma(2n+2\rho)\Gamma(n+2-\sigma)}{\Gamma(2-\sigma)\Gamma(n+2\rho)\Gamma(n+2)}}\,\mathrm{e}^{-\mathrm{i}\alpha e_n}. \tag{3.35}$$

Therefore using equations (3.22) and (3.35) in (2.14) we obtain

$$h_n(a_r) = \sqrt{\frac{\Gamma(2-\sigma)\Gamma(n+1)\Gamma(n+2)}{\Gamma(n+2-\sigma)}}\,\mathrm{e}^{\mathrm{i}\alpha e_n}, \tag{3.36}$$

and we can show that the normalization factor (2.15) in this case is given by

$$\mathcal{N}(|z|^2;a_r) = \left[\frac{1}{\Gamma(2-\sigma)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2-\sigma)}{\Gamma(n+2)}\frac{|z|^{2n}}{n!}\right]^{-1/2} = \frac{1}{\sqrt{\Phi(2-\sigma;2;|z|^2)}}, \tag{3.37}$$

where $\Phi(a;b;x) = {}_1F_1(a;b;x)$ is the degenerate hypergeometric function [32]. The coherent state of equation (2.13) obtained with these results is

$$|z;a_r\rangle = \frac{1}{\sqrt{\Gamma(2-\sigma)\Phi(2-\sigma;2;|z|^2)}}\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(n+2-\sigma)}{\Gamma(n+2)\Gamma(n+1)}}\,\mathrm{e}^{-\mathrm{i}\alpha e_n}z^n|\Psi_n\rangle. \tag{3.38}$$

With the help of the integral [33]

$$\int_0^{\infty} t^{\lambda-1}\,\mathrm{e}^{-t/2}\mathrm{W}_{\sigma,\mu}(t)\,\mathrm{d}t = \frac{\Gamma\left(\lambda-\mu-\frac{1}{2}\right)\Gamma\left(\lambda+\mu-\frac{1}{2}\right)}{\Gamma(\lambda-\sigma+1)}, \tag{3.39}$$

it is possible to show that the resolution of the unity can be obtained with the measure

$$w(|z|^2;a_r) = \frac{\Gamma(2-\sigma)}{\pi}\,\mathrm{e}^{-(|z|^2/2)}\Phi(2-\sigma;2;|z|^2)\mathrm{W}_{\sigma,1/2}(|z|^2), \tag{3.40}$$

where $\mathrm{W}_{\sigma,\mu}(x)$ is the Whittaker function [32, 33].

One example of the shape-invariant systems of type (B) is the Morse potential [34]. This potential has a finite number of normalizable bound states which cannot form a complete set of states in the Hilbert space, the condition necessary to construct the coherent state using our generalized approach. We nevertheless observe that since the superpotential $W_{\mathrm{B}}(x,a_1)$ has a special form (x-independent and linear in a_1-term) it is possible to construct coherent states for Morse potential systems using other sets of eigenstates that form a complete orthonormal basis in Hilbert space, and examples of these procedures can be found in [35, 36].

On the other hand, the shape-invariant systems classified as types (E) and (F) have superpotentials given by

$$W_{\mathrm{E}}(x,a_1) = \sqrt{\hbar\Omega}\left(\beta a_1\cot[\beta(x+\lambda)] + \frac{\delta}{a_1}\right) \qquad \text{and} \qquad W_{\mathrm{F}}(x,a_1) = \sqrt{\hbar\Omega}\left(\frac{a_1}{x} + \frac{\delta}{a_1}\right), \tag{3.41}$$

where β, δ and λ are real constants. The systems classified as type (E) only have bound states while the system type (F) have continuous as well as bound states. Like the systems of type (B), the (E) systems have a finite number of energy eigenstates. In this case, an alternative way is to construct the coherent state using the finite set of energy eigenstates with an adequate redefinition of the measure $w(|z|^2; a_r)$ to get a finite number of moments $\{\rho_n\}$, as was done in [37] for the Morse potential. The system of type (F), i.e. the Coulomb potential, because of its three-dimensional character, presents energy-degenerated eigenstates. In this case expansion (2.10) defined for our generalized coherent state must be appropriately adjusted for this situation. Our generalized definition (2.9) of coherent states for shape-invariant systems can be extended to include these alternative approaches for systems of types (B), (E) and (F). More details and further developments on this subject will be published elsewhere.

3.3. Self-similar potential systems

All previous examples have partner potentials $V_\pm(x)$ with parameters related by a translation. One class of shape-invariant potentials is given by an infinite chain of reflectionless potentials $V_\pm^{(k)}(x)$ $(k = 0, 1, 2, \ldots)$, for which associated superpotentials $W_k(x)$ satisfy the self-similar ansatz $W_k(x) = q^k W(q^k x)$, with $0 < q < 1$. These sets of partner potentials $V_\pm^{(k)}(x)$, also called self-similar potentials [38, 39], have an infinite number of bound states and parameters related by a scaling: $a_n = q^{n-1}a_1$. Shape invariance of self-similar potentials was studied in detail in [40, 41]. In the simplest case studied by them the remainder of equation (2.1) is given by $R(a_1) = ca_1$, where c is a constant. Hence

$$\prod_{k=1}^{n}\left[\sum_{s=k}^{n} R(a_s)\right] = \left[\frac{R(a_1)}{1-q}\right]^n q^{n(n-1)/2}(q;q)_n \tag{3.42}$$

where the q-shifted factorial $(q; q)_n$ is defined as $(p; q)_0 = 1$ and $(p; q)_n = \prod_{j=0}^{n-1}(1 - pq^j)$, with $n \in \mathbb{Z}$. Coherent states for self-similar potentials were introduced in [10, 13, 14]. Before applying our generalized approach for this system let us first assume the choice $\mathcal{Z}_j = 1$ and use it and the result of equations (3.42) in the expansion coefficient (2.14) to show that the coherent state (2.13) in this case is given by

$$|z; a_r\rangle = \frac{1}{\sqrt{E_q^{(-1/2)}(|\xi_0|^2)}}\sum_{n=0}^{\infty}\frac{q^{-n^2/4}}{\sqrt{(q;q)_n}}\xi_0^n|\Psi_n\rangle. \tag{3.43}$$

where $\xi_0 = z\sqrt{(1-q)/[\sqrt{q}R(a_1)]}$ and the q-exponential is defined by [42–44]

$$E_q^{(\mu)}(x) = \sum_{n=0}^{\infty}\frac{q^{\mu n^2}}{(q;q)_n}x^n. \tag{3.44}$$

The result (3.43) is the normalized form of the initial expression we obtained in our previous paper [13] for the coherent states of the self-similar potentials. To apply our generalized approach for this kind of potential system we assume

$$\mathcal{Z}_j = R(a_1)\,e^{-i\alpha R(a_1)} \qquad \text{yielding} \qquad \prod_{k=0}^{n-1}\mathcal{Z}_{j+k} = [R(a_1)]^n q^{n(n-1)/2}\,e^{-i\alpha e_n}, \tag{3.45}$$

where $e_n = R(a_1)(1 - q^n)/(1 - q)$. Substituting equations (3.42) and (3.45) in (2.14) we find

$$h_n(a_r) = \sqrt{\frac{(q;q)_n}{[R(a_1)(1-q)]^n q^{n(n-1)/2}}}\,e^{i\alpha e_n}, \qquad \text{and} \qquad \mathcal{N}(|z|^2; a_r) = \frac{1}{\sqrt{E_q^{(1/2)}(|\xi_1|^2)}}, \tag{3.46}$$

where $\xi_1 = z\sqrt{R(a_1)(1-q)/\sqrt{q}}$. The coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = \frac{1}{\sqrt{E_q^{(1/2)}(|\xi_1|^2)}} \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{\sqrt{(q;q)_n}} \, \mathrm{e}^{-\mathrm{i}\alpha e_n} \xi_1^n |\Psi_n\rangle. \tag{3.47}$$

In this case we can show that the inner product (2.17) of two coherent states can be readily found as

$$\langle z'; a_r | z; a_r\rangle = \frac{E_q^{(1/2)}(\xi_1'^* \xi_1)}{\sqrt{E_q^{(1/2)}(|\xi_1'|^2) E_q^{(1/2)}(|\xi_1|^2)}}. \tag{3.48}$$

This result is still valid for the first expression obtained for the coherent state of self-similar potentials (3.43) since we change $\xi_1 \to \xi_0$ and $E_q^{(1/2)}(x) \to E_q^{(-1/2)}(x)$. Equation (3.47) is the temporally stable version of the coherent state found in our previous paper [14] (see also [45]). As shown in that paper, this choice for $\mathcal{Z}_j$ makes it possible to establish an overcompleteness relation for the coherent state $|z; a_r\rangle$ using Ramanujan's integral extension of the beta function [46] and the measure, in this case, is given by

$$w(|z|^2; a_r) = \frac{1}{2\pi(-|\xi_1|^2; q)_\infty \log(1/q)}. \tag{3.49}$$

Finally, we introduce a possible new coherent state for this class of shape-invariant potentials by using the functional definition

$$\mathcal{Z}_j = R(a_1)\sqrt{1 - c/a_2}\, \mathrm{e}^{-\mathrm{i}\alpha R(a_1)} \quad \Longrightarrow$$
$$\prod_{k=0}^{n-1} \mathcal{Z}_{j+k} = [R(a_1)]^n q^{n(n-1)/2} \sqrt{\frac{(c;q^{-1})_{n+1}}{1-c}}\, \mathrm{e}^{-\mathrm{i}\alpha e_n}, \tag{3.50}$$

where c is an arbitrary constant. Substituting equations (3.42) and (3.50) in (2.14) we find

$$h_n(a_r) = \sqrt{\frac{(1-c)(q;q)_n}{[R(a_1)(1-q)]^n q^{n(n-1)/2} (c;q^{-1})_{n+1}}}\, \mathrm{e}^{\mathrm{i}\alpha e_n}, \qquad \text{and}$$
$$\mathcal{N}(|z|^2; a_r) = \sqrt{\frac{(-c|\xi_2|^2 q^{-1}; q)_\infty}{(-|\xi_2|^2; q)_\infty}}, \tag{3.51}$$

where $\xi_2 = z\sqrt{R(a_1)(1-q)}$. The coherent state (2.13) obtained with these results is

$$|z; a_r\rangle = \sqrt{\frac{(-c|\xi_2|^2 q^{-1}; q)_\infty}{(1-c)(-|\xi_2|^2; q)_\infty}} \sum_{n=0}^{\infty} q^{n(n-1)/4} \sqrt{\frac{(c;q^{-1})_{n+1}}{(q;q)_n}}\, \mathrm{e}^{-\mathrm{i}\alpha e_n} \xi_2^n |\Psi_n\rangle. \tag{3.52}$$

In this case it is possible to establish an overcompleteness relation for the coherent state $|z; a_r\rangle$ with the introduction of the measure

$$w(|\xi_2|^2; a_r) = \frac{R(a_1)(1-q)(1-c)}{\pi q \log(1/q)} \left[\frac{(-|\xi_2|^2; q)_\infty}{(-|\xi_2|^2 q^{-1}; q)_\infty}\right], \tag{3.53}$$

and using the Ramanujan integral given by [46]

$$\int_0^\infty t^{k-1} \frac{(-at; q)_\infty}{(-t; q)_\infty}\, \mathrm{d}t = \frac{\log(1/q)(q;q)_{k-1}}{q^{k(k-1)/2}(a;q^{-1})_k}. \tag{3.54}$$

(An elementary proof of (3.54) was given by Askey [47].) Note that it is straightforward to show that the results for this last example reduce to the previous one when we take the limit $c \to 0$ and consider the properties of the $(a;q)_\infty$ functions and the relations between the ξ_1

and ξ_2 complex variables. Indeed the Ramanujan integral and its integral extension of the beta function [46] used in the previous example is a particular case of the more general Ramanujan integral (3.54).

4. Concluding remarks

In this paper, using an algebraic approach, we constructed generalized coherent states for shape-invariant systems. This generalization based on the introduction of a factor functional $\mathcal{Z}_j$ of the potential parameters in the coherent state (a) satisfies the set of essential requirements we enumerated in the introduction to establish classical and quantum correspondence, (b) reproduces results already known for shape-invariant potential systems and (c) gives new possible expressions for coherent states. Another aspect to emphasize is that our generalized construction of coherent states gives some insight into the question of relating different sets of coherent states found in the literature for such systems.

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